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ON C-DELTA INTEGRALS ON TIME SCALES

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Abstract. In this paper we introduce the C-delta integral which generalize the McShane delta integral and investigate some properties of these integrals.

1. Introduction and preliminaries

The calculus on time scales was introduced for the first time in 1988 by Hilger[5] to unify the theory of difference equations and the theory of differential equations. It has been extensively studied on various aspects by several authors [2-4,5]. Surprisingly enough, the McShane integral has not received attention in the literature of time scales. In 2012, D. Jhao and X. You [14] introduced the McShane integral on time scales and some properties of this integral were studied. We introduce the C-delta integral which generalize the McShane delta integral and investigate some properties of these integrals.

A time scale $\mathbb{T}$ is a nonempty closed subset of real number $\mathbb{R}$ with the subspace topology inherited from the standard topology of $\mathbb{R}$. For $t \in \mathbb{T}$ we define the forward jump operator $\sigma(t)$ by $\sigma(t) = inf\{s > t : s \in \mathbb{T}\}$ where $inf\emptyset = sup\{\mathbb{T}\}$, while the backward jump operator $\rho(t)$ is defined by $\rho(t) = sup\{s < t : s \in \mathbb{T}\}$ where $sup\emptyset = inf\{\mathbb{T}\}$. If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. If $\sigma(t) = t$, we say that t is right-dense, while if $\rho(t) = t$, we say that t is left-dense. The forward graininess function $\mu(t)$ of $t \in \mathbb{T}$ is defined by $\mu(t) = \sigma(t) - t$, while the backward graininess function $\nu(t)$ of $t \in \mathbb{T}$

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is defined by $\nu(t) = t - \rho(t)$. For $a, b \in \mathbb{T}$ we denote the closed interval $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}$.

Throughout this paper, all considered intervals will be intervals in $\mathbb{T}$. A partition $\mathcal{D}$ of $[a, b]_{\mathbb{T}}$ is a finite collection of interval-point pairs $\{([t_{i-1}, t_i]_{\mathbb{T}}, \xi_i)\}_{i=1}^n$, where $\{a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b\}$ and $\xi_i \in [a, b]_{\mathbb{T}}$ for $i = 1, 2, \cdots, n$. By $\Delta t_i = t_i - t_{i-1}$ we denote the length of the ith subinterval in the partition $\mathcal{D}$. $\delta(\xi) = (\delta_L(\xi), \delta_R(\xi))$ is a Δ-gauge of $[a, b]_{\mathbb{T}}$ provided $\delta_L(\xi) > 0$ on $(a, b]_{\mathbb{T}}$, $\delta_R(\xi) > 0$ on $[a, b)_{\mathbb{T}}$, $\delta_L(a) \geq 0, \delta_R(b) \geq 0$ and $\delta_R(\xi) \geq \mu(\xi)$ for all $\xi \in [a, b)_{\mathbb{T}}$. we say that $\mathcal{D} = \{([t_{i-1}, t_i]_{\mathbb{T}}, \xi_i)\}_{i=1}^n$ is

(1) δ- fine McShane partition of $[a, b]_{\mathbb{T}}$ if $[t_{i-1}, t_i]_{\mathbb{T}} \subset [\xi_i - \delta_L(\xi_i), \xi_i + \delta_R(\xi_i)]_{\mathbb{T}}$ and $\xi_i \in [a, b]_{\mathbb{T}}$ for all $i = 1, 2, \cdots, n$.

(2) δ -fine Henstock partition of $[a, b]_{\mathbb{T}}$ if it is a δ-fine McShane partition of $[a, b]_{\mathbb{T}}$ and satisfying $\xi_i \in [t_{i-1}, t_i]_{\mathbb{T}}$.

(3) δ- fine C- partition of $[a, b]_{\mathbb{T}}$ if is a δ- fine McShane partition of $[a, b]_{\mathbb{T}}$ and satisfying the condition

$$\sum_{i=1}^n \operatorname{dist}([t_{i-1}, t_i]_{\mathbb{T}}, \xi_i) < \frac{1}{\epsilon},$$

where $\operatorname{dist}([t_{i-1}, t_i]_{\mathbb{T}}, \xi_i) = \inf\{|u_i - \xi_i| : u_i \in [t_{i-1}, t_i]_{\mathbb{T}}\}$.

Given a δ -fine C- partition (McShane partition) $\mathcal{D} = \{([t_{i-1}, t_i]_{\mathbb{T}}, \xi_i)\}_{i=1}^n$ we write

$$S(f, \mathcal{D}) = \sum_{i=1}^n f(\xi_i)(|[t_{i-1}, t_i]_{\mathbb{T}}|)$$

for integral sums over $\mathcal{D}$, whenever $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$.

Definition 1.1. [14] A function $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ is McShane delta integrable (McShane Δ-integrable) on $[a, b]_{\mathbb{T}}$ if there is a number A such that for each $\epsilon > 0$ there is a Δ-gauge, δ, of $[a, b]_{\mathbb{T}}$ such that

$$|S(f, \mathcal{D}) - A| < \epsilon$$

for each δ -fine McShane partition $\mathcal{D} = \{([t_{i-1}, t_i]_{\mathbb{T}}, \xi_i)\}_{i=1}^n$ of $[a, b]_{\mathbb{T}}$. A is called the McShane Delta integral of f on $[a, b]_{\mathbb{T}}$, and we write $A = (M) \int_a^b f(t)\Delta t$.

2. Definitions and basic properties of C-delta integral

Definition 2.1. A function $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ is C-delta integrable on $[a, b]_{\mathbb{T}}$ if there is a number A such that for each $\epsilon > 0$ there is a Δ-gauge,

δ, of $[a,b]_{\mathbb{T}}$ such that

$$|S(f,\mathcal{D}) - A| < \epsilon$$

for each δ -fine C- partition $\mathcal{D} = \{([t_{i-1},t_i]_{\mathbb{T}},\xi_i)\}_{i=1}^n$ of $[a,b]_{\mathbb{T}}$. A is called the C- delta integral of f on $[a,b]_{\mathbb{T}}$, and we write $A = \int_a^b f(t)\Delta t$. or $A = (C)\int_a^b f(t)\Delta t$.

By the definition of C-delta integral , McShane delta integral and Henstock delta integral, we get immediately the follwing theorem.

Theorem 2.2.
(1) *If f is McShane delta integrable on $[a,b]_{\mathbb{T}}$, then f is C- delta integrable on $[a,b]_{\mathbb{T}}$.*
(2) *If f is C- delta integrable on $[a,b]_{\mathbb{T}}$, then f is Henstock delta integrable on $[a,b]_{\mathbb{T}}$.*

Theorem 2.3. *A function $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is C-delta integrable on $[a,b]_{\mathbb{T}}$ if and only if for each $\epsilon > 0$ there is a Δ-gauge, δ, of $[a,b]_{\mathbb{T}}$ such that*

$$|S(f,\mathcal{D}_1) - S(f,\mathcal{D}_2)| < \epsilon$$

for any δ -fine C- partitions $\mathcal{D}_1$, $\mathcal{D}_2$ of $[a,b]_{\mathbb{T}}$.

Proof. Assume that $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is C-delta integrable on $[a,b]_{\mathbb{T}}$. For each $\epsilon > 0$ there is a Δ-gauge, δ, of $[a,b]_{\mathbb{T}}$ such that

$$|S(f,\mathcal{D}) - \int_a^b f(t)\Delta t| < \frac{\epsilon}{2}$$

for each δ -fine C- partition $\mathcal{D} = \{([t_{i-1},t_i]_{\mathbb{T}},\xi_i)\}_{i=1}^n$ of $[a,b]_{\mathbb{T}}$. If $\mathcal{D}_1$, $\mathcal{D}_2$ are δ -fine C- partitions of $[a,b]_{\mathbb{T}}$, then

$$|S(f,\mathcal{D}_1)-S(f,\mathcal{D}_2)| \le |S(f,\mathcal{D}_1)-\int_a^b f(t)\Delta t|+|S(f,\mathcal{D}_2)-\int_a^b f(t)\Delta t| < \epsilon.$$

Conversely, assume that for each $\epsilon > 0$ there is a Δ-gauge, δ, of $[a,b]_{\mathbb{T}}$ such that

$$|S(f,\mathcal{D}_1) - S(f,\mathcal{D}_2)| < \epsilon$$

for any δ -fine C- partitions $\mathcal{D}_1$, $\mathcal{D}_2$ of $[a,b]_{\mathbb{T}}$. Assume that $\{\delta_n\}$ is decreasing. For each $n \in N$, let $\mathcal{D}_n$ be a δ_n -fine C- partition of $[a,b]_{\mathbb{T}}$. Then $\{S(f,D_n)\}$ is a Cauchy sequence . Let $A = \lim_{n\to\infty} S(f,D_n)$ and let $\epsilon > 0$. Choose N such that $\frac{1}{N} < \frac{\epsilon}{2}$ and $|S(f,\mathcal{D}_n) - A| < \frac{\epsilon}{2}$ for all $n \ge N$. Let let $\mathcal{D}$ be a δ_N -fine C- partition of $[a,b]_{\mathbb{T}}$. Then

$$\begin{aligned} |S(f,\mathcal{D}_n)-A| &\le |S(f,\mathcal{D})-S(f,\mathcal{D}_N)|+|S(f,\mathcal{D}_N)-A| \\ &< \frac{1}{N}+\frac{\epsilon}{2}<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon>0. \end{aligned}$$

Hence, f is C-delta integrable on $[a,b]_{\mathbb{T}}$ and $A=\int_a^b f(t)\Delta t$. □

We can easily get the following theorems.

THEOREM 2.4. *Let $f:[a,b]_{\mathbb{T}}\to\mathbb{R}$ be a function. Then*
(1) *If f is C-delta integrable on $[a,b]_{\mathbb{T}}$, then f is C- delta integrable on every subinterval $[c,d]_{\mathbb{T}}$ of $[a,b]_{\mathbb{T}}$.*
(2) *If f is C-delta integrable on $[a,c]_{\mathbb{T}}$ and $[c,b]_{\mathbb{T}}$, then f is C-delta integrable on $[a,b]_{\mathbb{T}}$ and*

$$\int_a^b f(t)\Delta t=\int_a^c f(t)\Delta t+\int_c^b f(t)\Delta t.$$

THEOREM 2.5. *Let f and g be C-delta integrable functions on $[a,b]_{\mathbb{T}}$. Then*
(1) *kf is C-delta integrable on $[a,b]_{\mathbb{T}}$ and $\int_a^b kf(t)\Delta t=k\int_a^b f(t)\Delta t$ for each $k\in\mathbb{R}$.*
(2) *$f+g$ is C-delta integrable on $[a,b]_{\mathbb{T}}$ and $\int_a^b (f(t)+g(t))\Delta t=\int_a^b f(t)\Delta t$ $+\int_a^b g(t)\Delta t$.*

LEMMA 2.6. *(Saks-Henstock) Let $f:[a,b]_{\mathbb{T}}\to\mathbb{R}$ be C-Delta integrable on $[a,b]_{\mathbb{T}}$. Then for each ϵ there is a Δ-gauge, δ, of $[a,b]_{\mathbb{T}}$ such that*

$$|S(f,\mathcal{D})-\int_a^b f(t)\Delta t|<\epsilon$$

for each δ-fine C-partition $\mathcal{D}$ of $[a,b]_{\mathbb{T}}$. Particularly, if $\mathcal{D}'=\{([t_{i-1},t_i]_{\mathbb{T}},\xi_i)\}_{i=1}^m$ is an arbitrary δ-fine partial C-partition of $[a,b]_{\mathbb{T}}$, we have

$$|S(f,\mathcal{D}')-\sum_{i=1}^m\int_{t_{i-1}}^{t_i} f(t)\Delta t|\le\epsilon.$$

Proof. Assume that $\mathcal{D}'=\{([t_{i-1},t_i]_{\mathbb{T}},\xi_i)\}_{i=1}^m$ is an arbitrary δ-fine partial C-partition of $[a,b]_{\mathbb{T}}$. Let $\overline{[a,b]_{\mathbb{T}}-\cup_{i=1}^m[t_{i-1},t_i]_{\mathbb{T}}}=\cup_{j=1}^k[u_{j-1},u_j]_{\mathbb{T}}$.

Let $\eta>0$. Since f is C-delta integrable on each $[u_{j-1},u_j]_{\mathbb{T}}$, there is Δ-gauge, δ_j, of $[u_{j-1},u_j]_{\mathbb{T}}$ such that

$$|S(f,D_j)-\int_{u_{j-1}}^{u_j} f(t)\Delta t|<\frac{\eta}{\kappa}$$

for each δ_j-fine $C-$ partition D_j of $[u_{j-1}, u_j]_{\mathbb{T}}$.

Assume that $\delta_j(\xi) \leq \delta(\xi)$ for all $\xi \in [u_{j-1}, u_j]_{\mathbb{T}}$. Let $D_0 = D' \cup D_1 \cup \cdots \cup D_k$. Then D_0 is a δ-fine C-partition of $[a,b]_{\mathbb{T}}$ and we have

$$|S(f, D_0) - \int_a^b f(t)\Delta t| = |S(f, D') + \sum_{j=1}^{k} S(f, D_j) - \int_a^b f(t)\Delta t| < \epsilon$$

Consequently, we have

$$\begin{aligned}
& |S(f, D') - \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} f(t)\Delta t| \\
= \quad & |S(f, D_0) - \sum_{j=1}^{k} S(f, D_j) - (\int_a^b f(t)\Delta t - \sum_{j=1}^{k} \int_{u_{j-1}}^{u_j} f(t)\Delta t)| \\
\leq \quad & |S(f, D_0) - \int_a^b f(t)\Delta t| + \sum_{j=1}^{k} |S(f, [u_{j-1}, u_j]_{\mathbb{T}}) - \int_{u_{j-1}}^{u_j} f(t)\Delta t| \\
< \quad & \epsilon + k \cdot \frac{\eta}{k} = \epsilon + \eta.
\end{aligned}$$

Since $\eta > 0$ was arbitrary, we have $|S(f, D') - \sum_{i=1}^{m} \int_{I_i} f(t)\Delta t| \leq \epsilon$ $\square$

Lemma 2.7. *Let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be C- delta integrable on $[a,b]_{\mathbb{T}}$. Given $\epsilon > 0$, let δ be Δ-gauge function on $[a,b]_{\mathbb{T}}$ such that $|S(f, D) - \int_a^b f\Delta t| < \epsilon$ for each δ-fine C-partition $D = \{([t_{i-1}, t_i]_{\mathbb{T}}, \xi_i)\}_{i=1}^{m}$ of $[a,b]_{\mathbb{T}}$. If $D = \{([t_{i-1}, t_i]_{\mathbb{T}}, x_i)\}_{i=1}^{n}$ and $\{([u_{j-1}, u_j]_{\mathbb{T}}, y_j); 1 \leq j \leq m\}$ are δ-fine C-partitions of $[a,b]_{\mathbb{T}}$, then*

$$\sum_{i=1}^{n} \sum_{j=1}^{m} |f(x_i) - f(y_j)| \mu([t_{i-1}, t_i]_{\mathbb{T}} \cap [u_{j-1}, u_j]_{\mathbb{T}}) < 2\epsilon.$$

Proof. The nondegenerate intervals of the collection

$$\{[t_{i-1}, t_i]_{\mathbb{T}} \cap [u_{j-1}, u_j]_{\mathbb{T}} : 1 \leq i \leq n, 1 \leq j \leq m\}$$

form a partition of $[a,b]_{\mathbb{T}}$. Using these intervals, we form two C-partitions D_1 and D_2 of $[a,b]_{\mathbb{T}}$ as follows:

if $f(x_i) \geq f(y_j)$, then put $([t_{i-1}, t_i]_{\mathbb{T}} \cap [u_{j-1}, u_j]_{\mathbb{T}}, x_i) \in D_1$ and $([t_{i-1}, t_i]_{\mathbb{T}} \cap [u_{j-1}, u_j]_{\mathbb{T}}, y_j) \in D_2$.

if $f(x_i) < f(y_j)$, then put $([t_{i-1}, t_i]_{\mathbb{T}} \cap [u_{j-1}, u_j]_{\mathbb{T}}, y_j) \in D_1$ and $([t_{i-1}, t_i]_{\mathbb{T}} \cap [u_{j-1}, u_j]_{\mathbb{T}}, x_i) \in D_2$. Note that $S(f, D_1) - S(f, D_2) = \sum_{i=1}^{n} \sum_{j=1}^{m} |f(x_i) - f(y_j)| \mu([t_{i-1}, t_i]_{\mathbb{T}} \cap [u_{j-1}, u_j]_{\mathbb{T}})$.

Since D_1 and D_2 are δ-fine C-partition of $[a,b]_{\mathbb{T}}$,

$$\begin{aligned} S(f,D_1) &- S(f,D_2) \\ &\leq |S(f,D_1) - \int_a^b f(t)\Delta t| + |\int_a^b f(t)\Delta t - S(f,D_2)| \\ &\leq 2\epsilon \end{aligned}$$

□

THEOREM 2.8. *If $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is C-delta integrable on $[a,b]_{\mathbb{T}}$, then $|f|$ is C-delta integrable on $[a,b]_{\mathbb{T}}$.*

Proof. Let $\epsilon > 0$ and choose a Δ-gauge, δ, of $[a,b]_{\mathbb{T}}$ such that

$$|S(f,D) - \int_a^b f(t)\Delta t| < \frac{\epsilon}{2}$$

for each δ-fine C-partition $D = \{([t_{i-1},t_i]_{\mathbb{T}},\xi_i)\}_{i=1}^n$ of $[a,b]_{\mathbb{T}}$. We will show that $|f|$ satisfies the Cauchy criterion for C-delta integrability. Let $D_1 = \{[t_{i-1},t_i]_{\mathbb{T}},\xi_i)\}_{i=1}^n$ and $D_2 = \{([u_{j-1},u_j]_{\mathbb{T}},y_j)\}_{j=1}^m$ be δ-fine C-partitions of $[a,b]_{\mathbb{T}}$. Let $D_1' = \{([t_{i-1},t_i]_{\mathbb{T}} \cap [u_{j-1},u_j]_{\mathbb{T}},x_i) : 1 \leq i \leq n, 1 \leq j \leq m\}$ and $D_2' = \{([t_{i-1},t_i]_{\mathbb{T}} \cap [u_{j-1},u_j]_{\mathbb{T}},y_j) : 1 \leq i \leq n, 1 \leq j \leq m\}$, then D_1' and D_2' are δ-fine C-partitions of $[a,b]_{\mathbb{T}}$ and

$$\begin{aligned} S(|f|,D_1') &= \sum_{i=1}^n \sum_{j=1}^m |(f(x_i)|\mu([t_{i-1},t_i]_{\mathbb{T}} \cap [u_{j-1},u_j]_{\mathbb{T}}) \\ &= \sum_{i=1}^n |f(x_i)|\mu(I_i) = S(|f|,D_1) \\ S(|f|,D_2') &= \sum_{j=1}^m \sum_{i=1}^n |(f(y_i)|\mu([t_{i-1},t_i]_{\mathbb{T}} \cap [u_{j-1},u_j]_{\mathbb{T}}) \\ &= \sum_{j=1}^m |f(y_i)|\mu([u_{j-1},u_j]_{\mathbb{T}}) = S(|f|,D_2) \end{aligned}$$

Using the previous lemma, we obtain

$$\begin{aligned} &|S(|f|,D_1) - S(|f|,D_2)| = |S(|f|,D_1') - S(|f|,D_2')| \\ \leq &\sum_{i=1}^n \sum_{j=1}^m ||f(x_i)| - |f(y_j)||\mu([t_{i-1},t_i]_{\mathbb{T}} \cap [u_{j-1},u_j]_{\mathbb{T}}) \end{aligned}$$

$$\begin{aligned} &\leq \sum_{i=1}^{n}\sum_{j=1}^{m} |f(x_i) - f(y_j)|\mu(I_i \cap [u_{j-1}, u_j]_{\mathbb{T}}) \\ &< 2 \cdot \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Hence, the function $|f|$ is C-delta integrable on $[a,b]_{\mathbb{T}}$. □

DEFINITION 2.9. Let $F : [a,b]_{\mathbb{T}} \to \mathbb{R}$ and let E be a subset of $[a,b]_{\mathbb{T}}$.
(1) F is said to be AC_C on E if for each $\epsilon > 0$ there is a constant $\eta > 0$ and a Δ-gauge, δ, of $[a,b]_{\mathbb{T}}$ such that $|\sum_{i=1} F([t_{i-1}, t_i]_{\mathbb{T}})| < \epsilon$ for each δ -fine C- partition $\mathcal{D} = \{([t_{i-1}, t_i]_{\mathbb{T}}, \xi_i)\}_{i=1}^{n}$ of $[a,b]_{\mathbb{T}}$ and $\xi_i \in E$ and $\sum_{i=1} |[t_{i-1}, t_i]_{\mathbb{T}}| < \eta$.
(2) F is said to be ACG_C on E if F is continuous on E and E can be expressed as a countable union of sets on each of which F in AC_C.

THEOREM 2.10. *If a function $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is C-delta integrable on $[a,b]_{\mathbb{T}}$ with primitive F, then F is ACG_C on $[a,b]_{\mathbb{T}}$.*

Proof. By the definition of the C- delta integral and by the Saks-Henstock Lemma, F is continuous on $[a,b]_{\mathbb{T}}$ and for each $\epsilon > 0$ there is a Δ-gauge, δ, for $[a,b]_{\mathbb{T}}$ such that

$$|\sum_{i=1}^{n} [f(\xi_i)(|[t_{i-1}, t_i]_{\mathbb{T}}|) - F([t_{i-1}, t_i]_{\mathbb{T}})]| \leq \epsilon$$

for each δ -fine partial C- partition $\mathcal{D} = \{([t_{i-1}, t_i]_{\mathbb{T}}, \xi_i)\}_{i=1}^{n}$ of $[a,b]_{\mathbb{T}}$.

Assume that $E_n = \{\xi \in [a,b]_{\mathbb{T}} : n-1 \leq |f(\xi)| < n\}$ for each $n \in N$. Then we have $[a,b]_{\mathbb{T}} = \cup E_n$. To show that F is AC_C on each E_n, fix n and take a δ -fine C- partition $\mathcal{D}_0 = \{([t_{i-1}, t_i]_{\mathbb{T}}, \xi_i)\}_{i=1}^{n}$ of E_n $_{\mathbb{T}}$ and $\xi_i \in E_n$ for all i. If $\sum_{i=1}^{n} |[t_{i-1}, t_i]_{\mathbb{T}}| \leq \frac{\epsilon}{n}$, then

$$\begin{aligned} |\sum_{i=1} F([t_{i-1}, t_i]_{\mathbb{T}})| &\leq |\sum_{i=1} [F([t_{i-1}, t_i]_{\mathbb{T}} - f(\xi_i)|[t_{i-1}, t_i]_{\mathbb{T}}|]| \\ &\quad + \sum_{i=1} |f(\xi_i)|[t_{i-1}, t_i]_{\mathbb{T}}||[t_{i-1}, t_i]_{\mathbb{T}}| \\ &\leq \epsilon + n \sum_{i} |[t_{i-1}, t_i]_{\mathbb{T}}| < 2\epsilon. \end{aligned}$$

□

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